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The Predual of an Order-Unit Banach Space

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We give a shortened and simplified proof of the known theorem that a Banach space which is a predual of an order-unit Banach space is a base-normed space.

In 1964 Ellis [1] proved that a Banach space which is a predual of an order-unit-normed Banach space is, in fact, a base-normed space. The most interesting part of his proof is the use of a lemma he credits to Tukey and Klee, which is used to show the newly defined positive cone of the predual is generating and that the closure of the base-normed unit ball is the original closed unit ball of the predual. In this note we establish Ellis' result without using the Tukey–Klee lemma. We adapt another result of Klee, found in the book of Peressini [3, p. 194], which is quite simple compared with the original Tukey–Klee lemma, to accomplish the same outcome.

Let us introduce some notation and recall some elementary theorems about cones and dual cones. When (E, F) is a separated dual pair we use the theory of polars as developed in Kelley and Namioka [2]. If K is a cone in E, then $K^+ = \{f \in F\colon f(x) \geq 0 \text{ for all } x \in K\}$; further, $K^+{}_+ = \{x \in E\colon f(x) \geq 0 \text{ for all } f \in K^+\}$. If K were a cone in F, we could define K_+ and $K_+{}^+$ similarly. The cone K in E is called *generating* when $E = K - K$. Now, K^+ is a $w(F, E)$-closed cone in F when K is generating in E, in fact, K^+ is a cone *iff* $K - K$ is $w(E, F)$-dense in E [3, p. 71]. Also, $K^+{}_+$ is the $w(E, F)$-closure of K in E [3, p. 71].

E is said to be *Archimedean ordered* provided: x in E, $x \leq \alpha y$ for some y in K, and all $\alpha > 0$ in $\mathbb{R}$ implies $x \leq 0$ (x in $-K$). For a, b in E, $[a, b] = \{x \in E\colon a \leq x \leq b\}$. An element e in E is called an *order unit* if for each x in E there exists $\alpha > 0$ in $\mathbb{R}$ such that $x \in [-\alpha e, \alpha e]$, equivalently, $E =$

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$\cup_{n=1}^{\infty} n[-e, e]$. If E is Archimedean ordered and has an order unit e, then the Minkowski functional of $[-e, e]$ is a norm on E and E is said to be *order unit normed*. A nonempty convex set $B \subset K$ is called a *base* for K provided: for each nonzero x in K, there exist a unique $\alpha > 0$ in $\mathbb{R}$ and unique b in B such that $x = \alpha b$. When $S = con(B \cup -B)$ (the convex hull of $B \cup -B$) is linearly bounded the Minkowski functional of S is a norm on E and E is said to be *base normed*. We designate this base norm with $\|\cdot\|_B$ and observe that for each x in E: $\|x\|_B = \inf\{\alpha + \beta: x = \alpha\omega - \beta\nu, \omega, \nu \in B, \alpha, \beta \geq 0\}$.

Theorem. Suppose $(E, \|\cdot\|)$ is a Banach space and its dual E^* can be partially ordered so that its positive cone contains an order unit. If the norm of E^* is the order unit norm, then E can be partially ordered and its positive cone has a base, the cone is generating, and the base norm agrees with the original norm of E. Simply stated: A Banach predual of an order-unit Banach space is a base-normed space.

Proof. Let K be the positive cone of E^*, e the order unit in K, and $U = \{x \in E: \|x\| \leq 1\}$ the closed unit ball of E. Then, $U^\circ = [-e, e] = \{f \in E^*: -e \leq f \leq e\}$ is the closed unit ball of E^*; further, $U = [-e, e]_\circ = U^\circ_\circ$. Now, K is a generating cone in E^* (K contains an order unit); this implies K_+ is a $w(E, E^*)$-closed cone in E. The interval $[0, e]$ is $w(E^*, E)$-compact (Banach–Alaoglu Theorem [2, 17.4, p. 155]) and the Krein–Smulian Theorem [2, 22.6, p. 212] yields that $K = \cup_{n=0}^{\infty} n[0, e]$ is $w(E^*, E)$-closed. Thus ${K_+}^+$, being the $w(E^*, E)$-closure of K, must be K. It then follows that $G = K_+ - K_+$ is $w(E, E^*)$-dense in E. We now norm G as a base-normed space. If $x \in K_+$ and $x \neq 0$, then $e(x) \neq 0$, so $B = \{x/e(x): x > 0 \text{ in } E\}$ is a base for K_+ and the Minkowski functional on $S = con(B \cup -B)$, denoted $\|\cdot\|_B$, is a seminorm on G. First, $B \subset U$ ($x \geq 0$, $e(x) = 1 \Rightarrow -1 \leq f(x) \leq 1$ for all f in $[-e, e] = U^\circ \Rightarrow \|x\| \leq 1$) implies $S \subset U$. Thus, $\|\cdot\|_B$ is a norm and for each x in G, $\|x\|_B \geq \|x\|$. If $x \in B$, $e(x) = 1 \leq \|x\| \Rightarrow \|x\| = 1$; it then follows that $x \in K_+$, $x = \alpha b$, $b \in B$, $\alpha \geq 0$ implies $\|x\| = \alpha\|b\| = \alpha\|b\|_B = \|x\|_B$. So the two norms agree on K_+. We now show the two norms agree on G. This is accomplished by showing S is weakly dense in U, i.e., $U = S^\circ_\circ$. Since S U and $B^\circ = S^\circ$, we need only show $B^\circ \subset [-e, e]$. Let $f \in B^\circ \Rightarrow -1 \leq f(x/e(x)) \leq 1$ for all $x > 0$ in $E \Rightarrow -e(x) \leq f(x) \leq e(x)$ for all x in $K_+ \Rightarrow -e \leq f \leq e$ since $K = {K_+}^+$. Hence, $B^\circ \subset [-e, e]$ and S is weakly dense in U. Thus for each $x \in G$,

$$\|x\| = \sup\{f(x): f \in U^\circ = [-e, e]\}$$

Since $[-e, e] = S^\circ$,

$$\|x\|_B = \sup\{f(x): f \in S^\circ = \|x\|$$

Lastly we show G with the base norm is complete. Thus the cone K_+

is generating in E, the base norm is defined on E, and is in agreement with the original norm of E. Suppose $(z_n)_{n=1}^{\infty}$ is a base-normed Cauchy sequence in G such that $\|z_{n+1} - z_n\|_B < 1/2^n$. For each n, there exist scalars $\alpha_n, \beta_n \geq 0$ and vectors ω_n, ν_n in B for which $z_{n+1} - z_n = \alpha_n\omega_n - \beta_n\nu_n$ and $\alpha_n + \beta_n \leq 1/2^n$. For each $p \geq k$

$$\|\sum_{n=k}^{p} \alpha_n\omega_n\| = \|\sum_{n=k}^{p} \alpha_n\omega_n\|_B = \sum_{n=k}^{p} \alpha_n \leq \sum_{n=k}^{p} \alpha_n + \beta_n \leq \frac{1}{2^{k-1}}$$

So there exist $x, y \in K_+$ (K_+ is a complete cone) such that

$$x = \sum_{n=1}^{\infty} \alpha_n\omega_n, \qquad y = \sum_{n=1}^{\infty} \beta_n\nu_n$$

and these series converge in the normed topology on K_+. Consequently,

$$\lim_{n\to\infty} z_{n+1} - z_1 = \lim_{n\to\infty} \sum_{k=1}^{n} z_{k+1} - z_k = x - y \in G$$

Thus G is complete and must be E.

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